The Overlapping Plate Method for Asteroid Astrometry

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The overlapping plate method, first introduced in the early 1960s by Henrich Eichhorn, was originally intended to provide better star catalogs through improved data reduction techniques. Every star, not just the reference stars, contributes to the solution, and a group of overlapping exposures is processed in a simultaneous reduction. This method has become standard for star catalog developers, but only recently has it been applied to asteroid astrometry. Practical techniques for using the overlapping plate method are presented so that asteroid astrometrists can produce improved results.

Introduction

Classical reduction techniques in narrow-field astrometry (see, *e.g.*, König 1962) treat each exposure separately. One locates images of a set of reference stars, whose celestial coordinates are already known, and uses the measured coordinates of the images to determine the "plate constants" or set of parameters describing the ideal gnomonic projection and corrections to it. The plate constants can then be used to deduce the celestial coordinates of any other image in the field.

The classical approach works well provided that there is an adequate number of reference stars, well distributed around the target object, and that the target lies fairly close to the center of the field. The latter condition minimizes the effect of unmodeled or incorrectly modeled distortions on the derived position of the target (Eichhorn & Williams 1963). Since each field is reduced independently, a star whose image appears in several different fields taken on the same night will have a set of measured positions, all different, and one then calculates the weighted mean of these positions.

The "overlapping plate method" was introduced by Eichhorn (1960) specifically to handle cases where a star appeared on more than one photographic plate. This method relies on two principles:

- **A star can have only one position at a time.** One can solve directly for each star's position in a simultaneous solution, obtaining an answer that is better than any obtained from just one field.
- **One should use all available data.** Whenever a star appears more than once, there is a natural constraint on the relative pointing of the fields. In effect one synthesizes a larger field of view from all the exposures used in the reduction.

There are two advantages to this method. First, by moving the field around one can observe more reference stars. This can allow the observer to use a highly accurate but relatively sparse catalog like Tycho-2 instead of a denser but less accurate one. Second, those field stars which are observed more than once contribute to the plate constants for each field in which they appear. It thus becomes much easier to solve for distortions, magnitude terms, and other departures from the ideal projection. Both these effects make the solution more robust.

Note that a star that appears only once does not influence the plate constants. Rather, the solution for its position will drive its residual to zero, reproducing the results of the classical solution for the appropriate values of the plate constants for that field. This characteristic implies that, for asteroid work, one must treat each observation of the target asteroid as if it were a separate object.

The overlapping plate method has been used for several years at JPL's Table Mountain Observatory (Owen *et al.* 1998), with excellent results (Dunham 2000). It is presented here in the hope that software developers will consider including it in future releases of their astrometry packages.

Setup

Before one can use the overlapping plate method, one first needs an observation file containing not just the measured centroids but also catalogued positions of the reference stars and approximate positions of the field stars and of the targets at each observation. This file is obtained as follows:

1. Obtain centroids for every usable image in each picture.
2. Identify the reference stars and target(s) in the usual way.
3. Perform a classical astrometric solution on each field in turn. A six-constant solution may suffice.
4. Examine each centroid and write its data to the observation file:
   - If it is a reference star, identify it as such. Copy its coordinates and their uncertainties from the reference catalog.
   - If it is a target, identify it as such, and assign the observation a unique identifier (perhaps a concatenation of the asteroid number and the field number) so that each target image is treated independently of the others. Copy the astrometric, topocentric position from the target's ephemeris, or alternatively calculate (α, δ) from the plate constants. In either approach assign the position a "very large" (effectively infinite) uncertainty.
   - If it is a field star, calculate its (α, δ) and compare to the other field stars found thus far. Identify it with a previous observation if there is a match, or add it to the catalog if there is no match. Again use a very large uncertainty.
5. After the last image has been processed and identified, sort the observation file by "object," with all observations of each object placed together: reference stars first, then field stars, then the targets. An "object" in this context is a reference star, a field star, or *one observation* of a target.

The observation file contains one record for each observed image. Each record contains, at a minimum, the observed image location (x, y); the number of the field on which the image appeared; the type of image (reference star, field star, or target); an ID number unique to each "object"; the uncertainties σ_x and σ_y in the image centroid, and the uncertainties σ_α and σ_δ in the celestial coordinates. One can add other items as necessary, for instance the observed magnitude in order to add a coma term to the plate constants.

Note that this setup procedure need be executed but once. One can perform various solutions with different sets of solution parameters without having to recreate the observation file.

Note also that this procedure implicitly treats any new discoveries as though they were field stars. If the new asteroid is moving slowly enough that the matching algorithm can identify it as the same object on successive exposures, the solution will report one average position, and it will have huge residuals in each exposure. Conversely, if its motion between exposures exceeds the software's matching tolerance, it will be given a different ID at each appearance. It is a good idea to write the observation file as plain text so that one can fix this sort of problem manually.

The solution

The solution parameters include the right ascension α_i and declination δ_i for the ith object of the N objects, as well as the plate constants for all the fields. It is a good idea to solve for *changes* to these quantities: use the plate constants determined above to map (α_i, δ_i) into computed values (x_C, y_C), and subtract these from the observed (x, y) to form the prefit residuals. Thus the parameter set really looks like $(\Delta\alpha_1, \Delta\delta_1, \ldots, \Delta\alpha_N, \Delta\delta_N, \Delta A_1, \Delta B_1, \ldots)$. Of course, one may solve for $\Delta\alpha_i \cos\delta_i$ instead of $\Delta\alpha_i$.

The problem as it stands is singular: one can move the ensemble of fields (retaining their relative orientation) on the sky, simultaneously changing the coordinates of all the objects to compensate, and the observations will be unchanged. The singularity is removed by placing constraints on the positions of the reference stars. These constraints are nothing more than the catalogued uncertainties in the reference stars' positions, at the epoch of observation, and they amount to additional equations of condition of the form $\alpha_i - \alpha_{i_{\text{catalog}}} = 0 \pm \sigma_{\alpha_i}$.

This parameter set is very large, since it contains $2N$ parameters for the objects alone, as well as the plate constants for all the exposures. Computers nowadays can handle a least-squares adjustment with many thousands of parameters. Nevertheless, since each of the equations of condition includes terms for only one object, it is possible to use Kalman filtering and smoothing techniques (e.g., Bierman 1977) to shrink the size of the solution set. One replaces the first $2N$ parameters with two, a generic $\Delta\alpha$ (or $\Delta\alpha\cos\delta$) and a generic $\Delta\delta$. All the plate constants follow in the list. Data processing then proceeds in two steps: "filtering," in which the observations are presented and a final answer obtained; and "smoothing," in which the final answer for the plate constants is propagated back through the data to update the earlier partial results for the objects' coordinates.

Filtering

The filtering process is merely a mechanism for introducing the equations of condition sequentially into a least-squares adjustment.

1. Set up a covariance matrix whose $(1,1)$ element is $\sigma^2_{\alpha_1}$, whose $(2,2)$ element is $\sigma^2_{\delta_1}$, whose other diagonal elements are very large, and whose off-diagonal elements are zero. In other words, by supplying an *a priori* variance for α_1 and δ_1 one automatically incorporates the additional equations of condition described above. The other parameters are initially unconstrained and uncorrelated.

2. Process the equations of condition for each object in turn. Each image provides two such equations, one in x and one in y. The left-hand side of each equation contains terms in α_i, δ_i, and the plate constants for the field in question. The right-hand side is the prefit residual (observed minus computed value).

3. Each time a new object is encountered in the observation file:

    a. Save the current (interim) solution vector $\Delta\mathbf{q}_i^*$ to a scratch file.

    b. Calculate the first row of the inverse of the covariance matrix, and divide this row by its first element. Save this vector to the scratch file.

    c. Zero out the first row and first column of the covariance matrix, and set the $(1,1)$ element equal to the variance in right ascension for the next star, $\sigma^2_{\alpha(i+1)}$.

    d. Calculate the second row of the inverse of the (new) covariance matrix. Save this vector too.

e. Zero out the second row and column of the covariance matrix, and set the $(2,2)$ element equal to the variance in declination for the next star, $\sigma^2_{\delta_{(i+1)}}$.

The solution $\Delta\mathbf{q}_N$ after all N objects have been processed is the same as would be obtained in one huge least-squares adjustment. (Since it is the final answer rather than partial results, the asterisk is suppressed in the notation.) Likewise, the covariance is the same as the appropriate submatrix of the huge adjustment. The saved solution for each object (except for the last), however, is not the correct estimate; rather, it is what the correct estimate would be if there were no more observations to process. In order to obtain the correct estimate, one must work backwards through the data, in effect picking up each interim solution as it existed at the time and accounting for the effects of the subsequent observations.

Smoothing

The smoothing process (Rauch, Tung, & Streibel 1965; Bierman 1983) is the mechanism by which later observations are used to correct the interim solutions saved during filtering. For the ith object, there are no further equations of condition involving α_i and δ_i after the last observation of that object has been processed. Subsequent observations continue to adjust the plate constants, and changes in the plate constants induce compensating changes to (α_i, δ_i) through their correlations *as they were after the observations for object i were completed.* This is the reason the first two rows of the inverse of the covariance were saved.

Start at the end of the scratch file that was written during filtering, and proceed backwards through that file. The steps that follow assume that object i has been processed. (Initially $i = N$.)

1. Compute the postfit residuals for all observations of object i using the final (smoothed) solution $\Delta\mathbf{q}_i$.

2. Read in the previous three records: the interim solution $\Delta\mathbf{q}^*_{(i-1)}$ and the two rows of the inverse of the interim covariance.

3. Take the reciprocal of element 2 of the second (δ) row; this is the variance σ^{*2}_δ in declination for object $i-1$ for the correction we are about to make.

4. Multiply the second row by the above variance. The resulting vector, which has zero in its first element and unity in its second element, is the partial derivative vector $\partial\delta_{(i-1)}/\partial\mathbf{q}$. This vector is also known as the "smoother gains."

5. Perform a linear correction to obtained the smoothed correction in declination for object $i-1$:

$$\Delta\delta_{(i-1)} = \Delta\delta_i + \frac{\partial\delta_{(i-1)}}{\partial\mathbf{q}}(\Delta\mathbf{q}^*_{(i-1)} - \Delta\mathbf{q}_i).$$

   The second term is a product of a row vector (the partials) and a column vector (the corrections) and thus produces a scalar result. The left-hand side becomes element 2 of $\Delta\mathbf{q}_{(i-1)}$.

6. Optionally smooth the covariance by first treating the above partial derivatives and variance as if they constituted an equation of condition and then adding σ^{*2}_δ to the $(2,2)$ element of the covariance. This step provides the smoothed position sigmas for each object; it is independent of the smoothing process for $\Delta\mathbf{q}$.

7. Repeat steps 3-6 for right ascension (the first row). Then go back to step 1 for the postfit residuals for object $i-1$.

Factorization methods

It is worth noting that the classical Kalman filter is numerically unstable. Updates to the diagonal elements of the covariance are performed by a process which involves subtraction. When one datum provides an overwhelming decrease in the variance for a parameter, that subtraction can involve two nearly equal numbers, with resulting loss of precision. It is even possible for round-off and truncation errors to accumulate to the point where a diagonal element goes negative—and since the diagonal elements are the variances (σ^2) of the parameters, these elements must never be negative. One can imagine that havoc that ensues if this happens.

A far better way is to employ either the square root of the covariance or the square root of the inverse of the covariance. Both forms have been used at JPL; the astrometry reduction software happens to use the former. Bierman (1977) has developed numerically stable methods for both filtering and smoothing, and the interested reader is referred to his monograph for details. The stability arises because the diagonal elements are updated not by subtraction but by multiplication by a ratio of positive numbers; the resulting σ^2 can never be negative.

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